Particles with Zero Mass and Particles with "Small" Mass*

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(Received 22 October 1962)

The irreducible representations of the inhomogeneous Lorentz group which describe, respectively, particles with and without mass are qualitatively different. Nevertheless, the zero-mass representations can be obtained from the representation with mass by letting the mass go to zero. Particles of sufficiently small mass or sufficiently high energy behave like particles of zero mass.

THE classification of the irreducible representations of the inhomogeneous Lorentz group and explicit realizations of physical interest are well known.[1–7] Representations for zero-mass and nonzero-mass particles are characterized by different invariants and are qualitatively distinct.

On physical grounds, one should expect that particles with energies large compared to their mass would be indistinguishable from zero-mass particles.[7a] In other words, one should be able to obtain the realizations of the zero-mass representations by continuous transition from the case of nonzero mass. The purpose of this paper is to exhibit this transition explicitly.

The formal limit $m\rightarrow 0$ in the usual realization, Eqs. (4) below, leads to a reducible representation. By a simple canonical transformation this realization is automatically reduced in the limit of vanishing mass.

The generators of the inhomogeneous Lorentz group are the space translations $\mathbf{P}$, the time displacement H, the space rotations $\mathbf{J}$, and the proper Lorentz transformation $\mathbf{K}$. Their commutation relations ($c=\hbar=1$) are

$$\begin{aligned}
&[P_i,P_j]=0,\\
&[P_i,H]=0,\\
&[J_i,H]=0,\\
&[J_i,J_j]=i\textstyle\sum_k \epsilon_{ijk}J_k,\\
&[J_i,P_j]=i\textstyle\sum_k \epsilon_{ijk}P_k,\\
&[J_i,K_j]=i\textstyle\sum_k \epsilon_{ijk}K_k,\\
&[H,K_j]=-iP_j,\\
&[K_i,K_j]=-i\textstyle\sum_k \epsilon_{ijk}J_k,\\
&[P_i,K_j]=-i\delta_{ij}H.
\end{aligned}\tag{1}$$

The states of a particle with mass m and spin s may be represented by the function $\psi(\mathbf{p},\mu)$ with the norm

$$\|\psi\|^2=\sum_\mu \int d\mathbf{p}\,|\psi(\mathbf{p},\mu)|^2,\tag{2}$$

where $\mathbf{p}$ is the momentum and μ is the 3 component of the spin ($-s\leqslant\mu\leqslant s$). A complete set of operators operating on these functions is given by $\mathbf{p}$, $\mathbf{x}$, and $\mathbf{s}$, where $\mathbf{x}$ is defined by

$$x_k=i\partial/\partial p_k \quad (k=1,\cdots,3).\tag{3}$$

The spin $\mathbf{s}$ commutes with both $\mathbf{x}$ and $\mathbf{p}$. The three components of $\mathbf{s}$ satisfy the usual commutation rules of angular momenta.

The generators of the Lorentz group are then realized as

$$\begin{aligned}
&\mathbf{P}=\mathbf{p},\\
&H=(p^2+m^2)^{1/2}\equiv\omega,\\
&\mathbf{J}=\mathbf{x}\times\mathbf{p}+\mathbf{s},\\
&\mathbf{K}=\tfrac{1}{2}(\mathbf{x}\omega+\omega\mathbf{x})-(\mathbf{s}\times\mathbf{p})/(m+\omega),
\end{aligned}\tag{4}$$

where $p=|\mathbf{p}|$. The operator $\mathbf{x}$ is the position operator of Newton and Wigner.[8] For the following discussion, it is useful to write the last line in the form

$$\mathbf{K}=\tfrac{1}{2}(\mathbf{x}\omega+\omega\mathbf{x})-[(\mathbf{s}\times\mathbf{p})/p^2](\omega-m).\tag{5}$$

It is easy to verify that $[\mathbf{x}-(\mathbf{s}\times\mathbf{p})/p^2]$ and $(\mathbf{s}\cdot\mathbf{p})/p$ commute. Therefore, $(\mathbf{s}\cdot\mathbf{p})/p$ commutes with all generators of the group if $m=0$. The resulting representation is now reducible except for $s=0$. If space reflections are included, the representation for $s=\frac{1}{2}$ is still irreducible. Since $\mathbf{x}$ and $\mathbf{s}$ do not commute with $(\mathbf{s}\cdot\mathbf{p})/p$, a complete set of operators on the invariant subspaces is not immediately apparent. It is apparent, however, how we may obtain an equivalent representation that is already reduced in the limit of vanishing mass: We must diagonalize $(\mathbf{s}\cdot\mathbf{p})/p$.

Let U be a unitary operator which commutes with $\mathbf{P}$ and transforms $\mathbf{s}\cdot\mathbf{p}$ according to

$$U^\dagger(\mathbf{s}\cdot\mathbf{p})U=ps_3.\tag{6}$$

The operator

$$U=e^{-is_3\phi}e^{-is_2\theta}\tag{7}$$

* Work performed under the auspices of the U. S. Atomic Energy Commission.
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[1] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[2] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948).
[3] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).
[4] L. L. Foldy, Phys. Rev. **102**, 568 (1956).
[5] Iu. M. Shirokov, Zh. Eksperim. i Teor. Fiz. **33**, 861, 1196, 1208 (1957); **34**, 717 (1957); **36**, 879 (1959) [translations: Soviet Phys.—JETP **6**, 664, 919, 929 (1958); **7**, 493 (1958); **9**, 620 (1959)].
[6] M. A. Melvin, Rev. Mod. Phys. **32**, 477 (1960).
[7] J. S. Lomont and H. E. Moses, J. Math. Phys. **3**, 405 (1962).
[7a] E. P. Wigner, Rev. Mod. Phys. **29**, 255 (1957). See p. 257.
[8] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

has this property if ϕ and θ are the polar angles of $\mathbf{p}$ so that

$$p_1 = p\cos\phi\sin\theta,\quad p_2 = p\sin\phi\sin\theta,\quad p_3 = p\cos\theta. \tag{8}$$

Verification of the relation (6) is straightforward. It is useful to introduce a system of orthogonal unit vectors defined by

$$\mathbf{u}_1 = (\mathbf{p}\times\mathbf{n})/|\mathbf{p}\times\mathbf{n}| = (\sin\phi, -\cos\phi, 0),$$

$$\mathbf{u}_2 = \mathbf{p}\times(\mathbf{p}\times\mathbf{n})/p|\mathbf{p}\times\mathbf{n}| = (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta), \tag{9}$$

$$\mathbf{u}_3 = \mathbf{p}/p = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta),$$

where $\mathbf{n}$ is a unit vector in the direction of the 3 axis. From the commutation relations of $\mathbf{s}$, it follows that

$$U^\dagger\mathbf{s}U = s_1\mathbf{u}_2 - s_2\mathbf{u}_1 + s_3\mathbf{u}_3, \tag{10}$$

and, hence,

$$U^\dagger\mathbf{s}\cdot\mathbf{u}_3 U = s_3, \tag{11}$$

and

$$U^\dagger(\mathbf{s}\times\mathbf{u}_3)U = s_1\mathbf{u}_1 + s_2\mathbf{u}_2. \tag{12}$$

From (8) it follows that

$$-i[\mathbf{x},\theta] = (1/p)\mathbf{u}_2, \tag{13}$$

and

$$-i[\mathbf{x},\phi] = -(1/p\sin\theta)\mathbf{u}_1.$$

Hence,

$$U^\dagger\mathbf{x}U = \mathbf{x} + (s_1\mathbf{u}_1 + s_2\mathbf{u}_2 - s_3\mathbf{u}_1\cot\theta)/p. \tag{14}$$

From (11), (12), and (14) one can easily obtain a new set of variables $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{s}'$ which satisfy the same commutation relations as the old ones. They are defined by

$$s_k' = \mathbf{u}_k\cdot(\mathbf{s}\times\mathbf{u}_3),\quad k=1,2,\qquad s_3' = \mathbf{s}\cdot\mathbf{u}_3, \tag{15}$$

and

$$\mathbf{q} = \mathbf{x} - [\mathbf{s}\times\mathbf{u}_3 - (\mathbf{s}\cdot\mathbf{u}_3)\mathbf{u}_1\cot\theta]/p. \tag{16}$$

They satisfy the same commutation relations as $\mathbf{x}$, $\mathbf{p}$, and $\mathbf{s}$ since

$$U^\dagger\mathbf{q}U = \mathbf{x},\quad U^\dagger\mathbf{p}U = \mathbf{p},\quad U^\dagger\mathbf{s}'U = \mathbf{s}. \tag{17}$$

If we represent states by functions of $\mathbf{p}$ and the eigenvalue of s_3', then q_k is represented by $i\partial/\partial p_k$. Written as functions of $\mathbf{q}$, $\mathbf{p}$, and s', the generators $\mathbf{J}$ and $\mathbf{K}$ have the form

$$\mathbf{J} = \mathbf{q}\times\mathbf{p} + s_3'(\mathbf{n}\times\mathbf{u}_1)/(\mathbf{n}\times\mathbf{u}_1)\cdot\mathbf{u}_3, \tag{18}$$

$$\mathbf{n} = (0,0,1),$$

$$(\mathbf{n}\times\mathbf{u}_1)/(\mathbf{n}\times\mathbf{u}_1)\cdot u_3 = (\cos\phi/\sin\theta, \sin\phi/\sin\theta, 0), \tag{19}$$

$$\mathbf{K} = \tfrac{1}{2}(\mathbf{q}\omega + \omega\mathbf{q}) - s_3'(\omega/p)\mathbf{u}_1\cot\theta + (m/p)(s_1'\mathbf{u}_1 + s_2'\mathbf{u}_2). \tag{20}$$

The expression (18) is identical with that given by Shirokov[5] for zero-mass particles. The right-hand side of (20) becomes identical to Shirokov's zero-mass realization for $m=0$. As a possible position operator, $\mathbf{q}$ has the objectionable feature that its commutation relations with $\mathbf{J}$ are not those of a vector. This is obvious from the dependence of $\mathbf{q}$ on $\mathbf{n}$.

The following conclusions can be drawn from our results:

(1) Particles with "small" mass behave like particles with zero mass. (The mass m is "small" in this sense if it is small compared to all values of p for which the state function is appreciable.)

(2) High-energy particles ($p \gg m$) with definite helicity $(\mathbf{s}\cdot\mathbf{p})/p$ and different spins behave alike.

There still is, of course, the well-known "metaphysical" difference between particles of zero mass and particles of small mass. For zero-mass particles of helicity $\chi \geqslant 1$, the occurrence of otherwise identical particles with smaller helicity would be an accident. For particles of small mass one must expect such particles unless the interactions conspire not to produce them. For photons with small mass, this feature has been investigated extensively.[9–12]

Note added in proof. The limit $m \to 0$ in the representations of the inhomogeneous Lorentz group has also been investigated by Derek W. Robinson [Helv. Phys. Acta **35**, 98 (1962)]. The emphasis is on the continuous spin representations.

[9] L. de Broglie, *Mecanique Ondulatoire du Photon et Theorie Quantique des Champs* (Gauthier-Villars, Paris, 1949), Chap. 5.

[10] F. J. Belinfante, Progr. Theoret. Phys. (Kyoto) 4, 2 (1949).

[11] F. Coester, Phys. Rev. **83**, 798 (1951).

[12] M. M. Bass and E. Schrödinger, Proc. Roy. Soc. (London) **232**, 1 (1955).